

Bouncing Universes in Scalar-Tensor Gravity Around Conformal Invariance

B. Boisseau^{1*}, H. Giacomini^{1†}, D. Polarski^{2‡}

¹Université de Tours, Laboratoire de Mathématiques et Physique Théorique,
CNRS/UMR 7350, 37200 Tours, France

²Université Montpellier & CNRS, Laboratoire Charles Coulomb,
UMR 5221, F-34095 Montpellier, France

May 10, 2016

Abstract

We consider the possibility to produce a bouncing universe in the framework of scalar-tensor gravity when the scalar field has a nonconformal coupling to the Ricci scalar. We prove that bouncing universes regular in the future with essentially the same dynamics as for the conformal coupling case do exist when the coupling deviates slightly from it. This is found numerically for more substantial deviations as well. In some cases however new features are found like the ability of the system to leave the effective phantom regime.

PACS Numbers: 04.62.+v, 98.80.Cq

*email:bruno.boisseau@lmpt.univ-tours.fr

†email:hector.giacomini@lmpt.univ-tours.fr

‡email:david.polarski@umontpellier.fr

1 Introduction

Bouncing universes have attracted a lot of interest for various reasons. While on one hand they crucially modify the evolution of the universe by avoiding the initial singularity, they are also interesting to study from a mathematical point of view. A bouncing universe with a nonzero measure set of initial conditions can be obtained with a massive scalar field in a closed FLRW universe [1] where the curvature singularity is generically moved to the past. Non-singular solutions are degenerate (i.e. they exist only for a set of initial conditions which is of measure zero) [2], see also [3], while a bounce with positive spatial curvature requires severe fine tuning of initial conditions before the contraction stage [1], [4]. Spatially-flat FLRW non-degenerate bouncing universes have been built outside general relativity like theories with scalar [5, 6] or tensor ghosts, loop quantum gravity (see e.g. [7]) or gravity described by an effectively non-local Lagrangian (see also [8], [9], [10] and [11], [12] for recent reviews). In the framework of general relativity (GR) this requires the presence of some component of the phantom type around the bounce. In this sense interest in bouncing universes has an overlap with the field of dark energy (DE), where components of the phantom type are often considered. In particular, most modified gravity theories can produce an effective component of the phantom type, a property often put forward in favour of dark energy models outside general relativity (GR). Let us note that recent data do actually support the possibility of an effective DE component of the phantom type (see e.g. [13]). In this work we consider some general class of scalar-tensor theories of gravity, a well understood and widely studied contender of GR. It is known that these theories can violate the null-energy condition [14],[15],[16]. However, the construction of a concrete regular nondegenerate spatially flat bouncing model is a highly nontrivial challenge. It came therefore as a surprise that this could be achieved in a simple model with a conformally coupled scalar field and a negative potential [17]. Moreover the problem turned out to be completely integrable and exact analytical solutions were found. Translating this solution in the Einstein frame [18] gives an integrable minimally coupled scalar field with an inverted double-well potential, also found without relation to bouncing solutions in the Jordan frame in [19], while the viability bounds $F = 0$ correspond to either a Big Bang or a Big Crunch [18]. These bouncing solutions were extended to non-flat geometries in [20].

While its importance was clear regarding the possibility to derive exact solutions, the relevance of the conformal coupling for the existence and the dynamics of bouncing solutions remained unclear. The importance of conformal symmetry has been recognized and used in many physical problems (see e.g. [21], [22], [23], [24], [25]). This motivates us even more to assess its relevance here. Also, the conformally invariant bouncing model found rested on a background satisfying $\dot{H} > 0$ forever after the bounce, which can be seen as a problem if one is willing to build a realistic cosmological scenario with it. It is therefore

interesting to see whether a richer behaviour is obtained with a slight generalization of our model. We will consider minimal breaking of conformal invariance by studying a wider class of couplings (see [26] for a similar study with other motivations) while leaving the potential unchanged, and investigate in how far the bouncing solutions, if they still exist, depart from the bouncing solutions found earlier with conformal coupling.

2 The conformally invariant bouncing model

We consider a universe where gravity is described by a scalar-tensor model. The Lagrangian density in the Jordan frame of the gravitational sector is given by (see e.g. [27])

$$L = \frac{1}{2} (F(\Phi)R - Z(\Phi) g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2U(\Phi)) .$$

We can take $Z = 1$ or $Z = -1$, corresponding physically to $\omega_{BD} > 0$ or $\omega_{BD} < 0$. For $\omega_{BD} < 0$, the theory is ghost-free provided $-\frac{3}{2} < \omega_{BD} < 0$. We consider further spatially flat FLRW universes with metric $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$ yielding the following modified Friedmann equations

$$3FH^2 = \frac{1}{2}Z\dot{\Phi}^2 - 3H\dot{F} + U , \quad (1)$$

$$-2F\dot{H} = Z\dot{\Phi}^2 + \ddot{F} - H\dot{F} , \quad (2)$$

with $H \equiv \frac{\dot{a}}{a}$. Here and below a dot, resp. a prime, stands for the derivative with respect to t , resp. to Φ . Equations (1),(2) contain the equation of motion of Φ

$$Z(\ddot{\Phi} + 3H\dot{\Phi}) - 3F'(\dot{H} + 2H^2) + U' = 0 . \quad (3)$$

We study the specific model

$$ZF = -\xi\Phi^2 + \kappa^{-2} , \quad (4)$$

$$ZU = \frac{\Lambda}{\kappa^2} - c\Phi^4 . \quad (5)$$

The equations (1) – (3) then become

$$3(-\xi\Phi^2 + \kappa^{-2})H^2 = \frac{1}{2}\dot{\Phi}^2 + 6H\xi\Phi\dot{\Phi} + \Lambda\kappa^{-2} - c\Phi^4 , \quad (6)$$

$$-2(-\xi\Phi^2 + \kappa^{-2})\dot{H} = \dot{\Phi}^2 - 2\xi\dot{\Phi}^2 - 2\xi\Phi\ddot{\Phi} + 2H\xi\Phi\dot{\Phi} , \quad (7)$$

$$\ddot{\Phi} + 3H\dot{\Phi} + 6\xi\Phi(\dot{H} + 2H^2) - 4c\Phi^3 = 0 . \quad (8)$$

We emphasize that if we put all terms of eq.(6) on the same side we obtain a vanishing first integral of eq.(7)-(8). If $\xi = \frac{1}{6}$, we recover the system studied in [17] for which exact

bouncing universes regular in the future were found, and equation (3) corresponds to a massless scalar field conformally coupled to Einstein gravity with an interaction potential $-c\Phi^4$. We take $Z = 1$ because for $Z = -1$ only one initial condition $\Phi_0 \equiv \Phi_{0,cr}$ yields a viable solution. The solutions found were regular in the future in the usual sense that they do not diverge but also because they satisfy $F > 0$ at all times after passing the bounce. For convenience we locate the bounce at $t = 0$ with $H(t = 0) \equiv H_0 = 0$ (we will adopt a similar notation for quantities at $t = 0$). The allowed interval of initial values Φ_0 is given by $\Phi_{0,min} \leq \Phi_0 < \Phi_{max}$. The potential U vanishes at $\Phi_{0,min}$ while F vanishes at Φ_{max} . We note from (6) that the field velocity at the bounce is given up to a sign by the value of Φ_0 . A family of solutions with non-zero measure in the set of initial conditions was found with initial values between some critical value $\Phi_{0,cr}$ and Φ_{max} (and $\dot{\Phi}_0 < 0$) and tending asymptotically to zero in a monotonous way. Remarkably, $H(t)$ was found to couple to a first integral of Φ only. So it is the same function of time $H = \sqrt{\frac{\Lambda}{3}} \tanh \left[2\sqrt{\frac{\Lambda}{3}} t \right]$ irrespective of the specific dynamics of Φ . The unique solution starting precisely at $\Phi_{0,cr}$ tends asymptotically to some non-zero value $\tilde{\Phi} \equiv (\frac{\Lambda}{6c})^{\frac{1}{2}}$. All solutions starting at $t = 0$ in the interval $\Phi_{0,min} \leq \Phi_0 < \Phi_{0,cr}$ were found to diverge in a finite time. While conformal invariance is obviously lost for $\xi \neq \frac{1}{6}$, we will next show that bouncing solutions regular in the future exist in that case too.

3 Bouncing solutions for $\xi \neq \frac{1}{6}$

Let us first introduce the parameter ϵ defined as follows

$$\xi = \frac{1}{6}(1 + \epsilon) . \quad (9)$$

Clearly, $\epsilon = 0$ corresponds to conformal invariance. The condition $F > 0$ holds for $\Phi < \Phi_{max}$, viz.

$$\Phi < \Phi_{max} = \frac{\sqrt{6}}{\kappa} \frac{1}{\sqrt{1 + \epsilon}} , \quad Z = 1 . \quad (10)$$

Moreover, from (1) we must have at the bounce $\Phi_0 \geq \Phi_{0,min} = \left(\frac{\Lambda}{\kappa^2 c}\right)^{\frac{1}{4}}$. Note that $\Phi_{0,min}$ does not depend on ϵ . Therefore the existence of a nonvanishing interval of initial values leading to regular bouncing solutions requires

$$\sqrt{1 + \epsilon} < \Phi_{0,min}^{-1} \Phi_{max}^{(0)} \equiv \sqrt{1 + \epsilon_{max}} , \quad (11)$$

where we have defined $\Phi_{max}^{(0)} \equiv \Phi_{max}(\epsilon = 0)$, a useful notation that will be extended to all quantities. We have further introduced in (11) the quantity ϵ_{max}

$$\epsilon_{max} = \frac{6}{\kappa} \left(\frac{c}{\Lambda} \right)^{\frac{1}{2}} - 1 . \quad (12)$$

The initial conditions of our system are completely defined from Φ_0 . Indeed from (1) or (6), $\dot{\Phi}_0$ is fixed (up to a sign) once Φ_0 is given. Moreover for a given Φ_0 , the same value $\dot{\Phi}_0$ is obtained independently of ϵ . We will use this property later. It is also obvious that the allowed interval of initial values for $\epsilon > 0$ is smaller than the interval for $\epsilon = 0$ while it is larger for $\epsilon < 0$.

We write further equations (7), (8) in a form suitable for a dynamical system analysis

$$\dot{H} = -\frac{\kappa^2 ((2 - \epsilon)y^2 + 4(1 + \epsilon)Hy\Phi + 2(1 + \epsilon)^2\Phi^2H^2 - 4c(1 + \epsilon)\Phi^4)}{6 + \epsilon(1 + \epsilon)\kappa^2\Phi^2}, \quad (13)$$

$$\dot{\Phi} = y, \quad (14)$$

$$\begin{aligned} \dot{y} = & \frac{Hy(-18 + (4 + 5\epsilon + \epsilon^2)\kappa^2\Phi^2) + H^2\Phi(-12(1 + \epsilon) + 2(1 + \epsilon)^2\kappa^2\Phi^2)}{6 + \epsilon(1 + \epsilon)\kappa^2\Phi^2} \\ & + \frac{\Phi((2 + \epsilon - \epsilon^2)\kappa^2y^2 + 4c(6\Phi^2 - (1 + \epsilon)\kappa^2\Phi^4))}{6 + \epsilon(1 + \epsilon)\kappa^2\Phi^2}. \end{aligned} \quad (15)$$

Remember that (6) constrains the initial conditions. Once the equations are written this way, we can apply an important theorem about the existence of solutions in the neighbourhood of $\epsilon = 0$. Later on however we will return to the original form given in eq.(6) - (8).

Equations (13),(15) are well-defined if the denominator appearing in these equations does not vanish, viz.

$$6 + \epsilon(1 + \epsilon)\kappa^2\Phi^2 \neq 0. \quad (16)$$

Clearly, vanishing of the denominator is possible for $\epsilon < 0$ only with $\Phi \sim \frac{\sqrt{6}}{\kappa} \frac{1}{\sqrt{|\epsilon|}}$ for $\epsilon \rightarrow 0$. One can show (see e.g. [28] for a formal statement of the theorem) that in the region where the inequality (16) holds, there exists some time interval $-a \leq t \leq b$ with $a, b > 0$ where regular solutions $\Phi(t)$, $y(t)$ and $H(t)$ exist which are analytic functions of ϵ around $\epsilon = 0$. This time interval can be extended as long as the solution does not diverge and inequality (16) is satisfied.

As we show now, we can take $b \rightarrow \infty$ for ϵ sufficiently small. Indeed, using only continuity with respect to ϵ we can write for $t < b$

$$H(t) = H^{(0)}(t) + O(\epsilon), \quad (17)$$

$$\Phi(t) = \Phi^{(0)}(t) + O(\epsilon). \quad (18)$$

We use now the crucial property that (the regular solutions) $H^{(0)}$, $\Phi^{(0)}$ are bounded functions for all times $t > 0$ satisfying $\Phi^{(0)}(t) < \frac{\sqrt{6}}{\kappa}$, $H^{(0)}(t) < \sqrt{\frac{\Lambda}{3}}$. Now it is clear that in the neighbourhood of the zeroth order solution, the equations are regular as they satisfy (16) for all times $t > 0$ so that the theorem applies. In eq.(18) the initial value should be

a function of ϵ too with $\Phi_0 \rightarrow \Phi_0^{(0)}$. We will consider in the neighbourhood of a given $\Phi^{(0)}$ a family of solutions depending on ϵ with a fixed initial value Φ_0 . In this way the initial conditions at the bounce are independent of ϵ and the function $\Phi^{(0)}$ to which Φ tends for $\epsilon \rightarrow 0$ is completely defined. A subtlety arises here for $\epsilon < 0$ because some initial values leading to regular solutions start outside the allowed interval of initial values for $\epsilon = 0$. We have checked numerically that solutions starting there for $\epsilon = 0$ are regular indeed. These solutions were discarded not for mathematical but for physical reasons as they start with $F < 0$. But from a mathematical point of view, such solutions are perfectly admissible.

We would like to show further that the time interval where (17),(18) apply can be extended to infinity. Let us assume that $\Phi(t)$ (and/or $H(t)$) diverges at some time $b < \infty$. Then $\Phi(t)$ (and/or $H(t)$) could become arbitrarily large for $t \rightarrow b$ but this would clearly be in contradiction with (17),(18). So we have shown that (17)-(18) is valid also for t going to infinity. It is also obvious that in a neighbourhood around $\epsilon = 0$ the condition (10) is satisfied so that the solutions obtained are physically acceptable as well. Note that this does not preclude the existence of regular solutions for larger values of ϵ .

To summarize, we get a family of solutions $H(t)$ and $\Phi(t)$ arbitrarily close to the conformally invariant solutions $H^{(0)}$, $\Phi^{(0)}$. This means that conformal invariance, though crucial for the derivation of *exact analytical* expressions, is not needed in order to have regular bouncing solutions. Of course, it is a priori not clear how far one can depart from conformal invariance. This will be investigated later with numerical calculations. However before resorting to numerical calculations, we can gain further analytical insight by studying the behaviour of this family of solutions at each order in a systematic expansion in ϵ .

Like for $\epsilon = 0$, an analysis in the asymptotic regime [17], assuming only $H \rightarrow \text{constant}$, gives that the solution with initial value $\Phi_{0,cr}$ tends asymptotically to $\tilde{\Phi} = \tilde{\Phi}^{(0)} \sqrt{1 + \epsilon}$, with $\Phi_{0,cr}$ depending also on ϵ . From (6) we have $H(t) \rightarrow \sqrt{\frac{\Lambda}{3}}$ when $\Phi \rightarrow 0$ and also when $\Phi \rightarrow \tilde{\Phi}$. So all regular solutions tend asymptotically to a de Sitter space while gravity tends dynamically to GR.

4 Asymptotic behaviour at all order in ϵ

We want now to investigate the solutions obtained at each order in a systematic expansion in ϵ . In particular analytic expressions will be obtained at first order in the asymptotic regime $t \rightarrow \infty$. Using analyticity in ϵ , we write

$$H(t) = H^{(0)}(t) + \epsilon H^{(1)}(t) + O(\epsilon^2) , \quad (19)$$

$$\Phi(t) = \Phi^{(0)}(t) + \epsilon \Phi^{(1)}(t) + O(\epsilon^2) . \quad (20)$$

In a standard way, substituting these expressions in our basic equations (6)-(8) with the definition (9), we obtain at first order three linear differential equations for $H^{(1)}(t)$, $\Phi^{(1)}(t)$ with time-dependent coefficients depending on the zeroth order solutions $H^{(0)}$, $\Phi^{(0)}$. Equation (6) yields

$$\begin{aligned} & \left(-6\kappa^{-2}H^{(0)} + H^{(0)}\Phi^{(0)2} + \Phi^{(0)}\dot{\Phi}^{(0)} \right) H^{(1)} + \left(H^{(0)}\Phi^{(0)} + \dot{\Phi}^{(0)} \right) \dot{\Phi}^{(1)} + \\ & \left(H^{(0)2}\Phi^{(0)} - 4c\Phi^{(0)3} + H^{(0)}\dot{\Phi}^{(0)} \right) \Phi^{(1)} + 3H^{(0)2}\Phi^{(0)2} + 6H^{(0)}\Phi^{(0)}\dot{\Phi}^{(0)} = 0 . \end{aligned} \quad (21)$$

Proceeding in the same way we obtain from (7)

$$\begin{aligned} & \left(2\kappa^{-2} - \frac{\Phi^{(0)2}}{3} \right) \dot{H}^{(1)} - \frac{1}{3}\Phi^{(0)}\ddot{\Phi}^{(1)} + \frac{1}{3}\Phi^{(0)}\dot{\Phi}^{(0)}H^{(1)} \\ & + \frac{1}{3} \left(-2\Phi^{(0)}\dot{H}^{(0)} + H^{(0)}\dot{\Phi}^{(0)} - \ddot{\Phi}^{(0)} \right) \Phi^{(1)} + \left(\frac{1}{3}H^{(0)}\Phi^{(0)} + \frac{4}{3}\dot{\Phi}^{(0)} \right) \dot{\Phi}^{(1)} \\ & - 2 \left(\Phi^{(0)2}\dot{H}^{(0)} - H^{(0)}\Phi^{(0)}\dot{\Phi}^{(0)} + \dot{\Phi}^{(0)2} + \Phi^{(0)}\ddot{\Phi}^{(0)} \right) = 0 , \end{aligned} \quad (22)$$

and finally from (8)

$$\begin{aligned} & \ddot{\Phi}^{(1)} + 3H^{(0)}\dot{\Phi}^{(1)} + \left(2H^{(0)2} - 12c\Phi^{(0)2} + \dot{H}^{(0)} \right) \Phi^{(1)} = \\ & - \left(4H^{(0)}\Phi^{(0)} + 3\dot{\Phi}^{(0)} \right) H^{(1)} - \Phi^{(0)}\dot{H}^{(1)} - 6 \left(2H^{(0)2}\Phi^{(0)} + \Phi^{(0)}\dot{H}^{(0)} \right) . \end{aligned} \quad (23)$$

The right hand side of equation (21) is a first integral of (22), (23). As the coefficients of the highest derivatives in (22), (23) do not vanish, and none of the other coefficients are singular, the solutions of our linear system are regular. Using these results, we proceed with the study of the solutions at first order and investigate their behaviour in the asymptotic regime $t \rightarrow \infty$. So in what follows equations and solutions refer specifically to this regime. As was shown in [17], the family of lowest order solutions $\Phi^{(0)}$ tending to zero obey at leading order the equation

$$\ddot{\Phi}^{(0)} + \sqrt{3\Lambda}\dot{\Phi}^{(0)} + \frac{2\Lambda}{3}\Phi^{(0)} = 0 , \quad (24)$$

whose general solution is

$$\Phi^{(0)} \sim C_1 \exp \left(-\sqrt{\frac{\Lambda}{3}} t \right) + C_2 \exp \left(-2\sqrt{\frac{\Lambda}{3}} t \right) , \quad (25)$$

where C_1 , C_2 are constants. Remember that there exists another unique solution $\Phi^{(0)}$ tending asymptotically to the nonzero finite value $\tilde{\Phi}^{(0)}$. For $H^{(0)}(t)$ we have at leading order

$$H^{(0)} \sim \sqrt{\frac{\Lambda}{3}} , \quad (26)$$

$$\dot{H}^{(0)} \sim 8\frac{\Lambda}{3}\exp \left(-4\sqrt{\frac{\Lambda}{3}} t \right) . \quad (27)$$

Using (25), (26), we see immediately from (21) that we must have $H^{(1)} \rightarrow 0$ if $\Phi^{(1)}$ and $\dot{\Phi}^{(1)}$ remain bounded. Even if we do not assume a priori that $\Phi^{(1)}$ and $\dot{\Phi}^{(1)}$ remain bounded, we can use (21) to write

$$H^{(1)} = A^{(1)}(t) \dot{\Phi}^{(1)} + B^{(1)}(t) \Phi^{(1)} + D^{(1)}(t) , \quad A^{(1)}, B^{(1)}, D^{(1)} \rightarrow 0 . \quad (28)$$

Therefore in this regime, (23) becomes at leading order

$$\ddot{\Phi}^{(1)} + \sqrt{3\Lambda} \dot{\Phi}^{(1)} + \frac{2}{3}\Lambda\Phi^{(1)} = I^{(1)}(t) , \quad (29)$$

where the inhomogeneous part $I^{(1)}(t) = -4\Lambda\Phi^{(0)}$ decays exponentially. Clearly the inhomogeneous solution decays exponentially too as we will see explicitly. The homogeneous part of eq.(29) is precisely the asymptotic equation satisfied by $\Phi^{(0)}$. Though the inhomogeneous solution decays exponentially, it gives the solution of eq.(29) at leading order, namely

$$\Phi^{(1)} \sim -4C_1\sqrt{3\Lambda} t \exp\left(-\sqrt{\frac{\Lambda}{3}} t\right) . \quad (30)$$

As we have from (30) that $\Phi^{(1)}$ and $\dot{\Phi}^{(1)}$ vanish asymptotically, we know that $H^{(1)}$ goes to zero as mentioned earlier. We find again at leading order from (21) using (25)

$$H^{(1)} \sim -\frac{C_1^2\kappa^2}{2}\sqrt{\frac{\Lambda}{3}}\exp\left(-2\sqrt{\frac{\Lambda}{3}} t\right) . \quad (31)$$

Hence for $\epsilon \rightarrow 0$, $\epsilon > 0$, we have (asymptotically) $H \rightarrow H^{(0)}$ with $H < H^{(0)}$ and $\dot{H} > 0$. On the other hand for $\epsilon \rightarrow 0$, $\epsilon < 0$, we have $H \rightarrow H^{(0)}$ with $H > H^{(0)}$ and, using (27) and (31), $\dot{H} < 0$. The latter behaviour is different from that found in the conformally coupled case. This is in agreement with our numerical simulations (see below).

So we have derived explicit expressions for $\Phi^{(1)}$ and $H^{(1)}$ for $t \rightarrow \infty$. It is seen that $H^{(1)}$ tends much more rapidly to zero than $\Phi^{(1)}$. In particular for regular solutions, $H \rightarrow H^{(0)}$ much quicker than $\Phi \rightarrow \Phi^{(0)}$ for ϵ very small. Following the same reasoning at higher orders, it is easy to see that if $H^{(n-1)}$ and $\Phi^{(n-1)}$ tend to zero, then this will be also true for $H^{(n)}$ and $\Phi^{(n)}$. Indeed the constraint equation (21) yields at n th order

$$H^{(n)} = A^{(n)}(t) \dot{\Phi}^{(n)} + B^{(n)}(t) \Phi^{(n)} + D^{(n)}(t) , \quad A^{(n)}, B^{(n)}, D^{(n)} \rightarrow 0 . \quad (32)$$

Then the homogeneous equation satisfied by $\Phi^{(n)}$ will be again similar to (24) with a non-homogeneous part $I^{(n)}(t)$ decaying exponentially. To summarize, we have shown explicitly that $H^{(n)}$ and $\Phi^{(n)}$ tend exponentially to zero for $t \rightarrow \infty$ at all orders n .

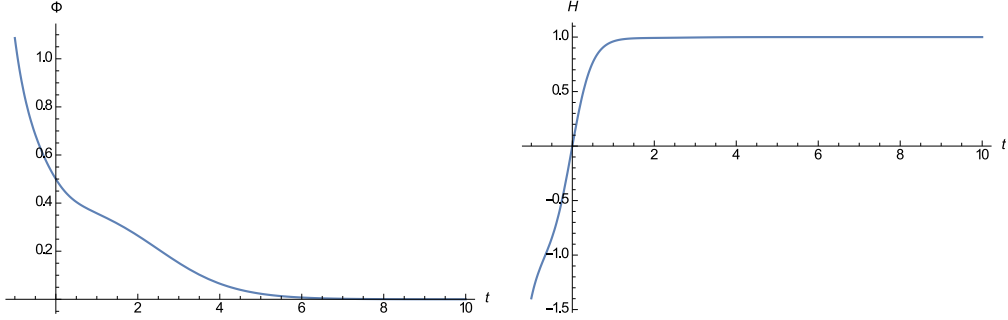


Figure 1: The functions $\Phi(t)$ and $H(t)$ are shown starting before the bounce (located at $t = 0$) for $\epsilon = 0.1$. The model parameters are $\kappa = \sqrt{20}$, $\Lambda = 3$ and $c = 3$. For this model we have $\Phi_{max}^{(0)} = 0.5477$, $\Phi_{0,min} = 0.4728$, $\epsilon_{max} = 0.3416$ and $\Phi_{max} = \Phi_{max}^{(0)} \frac{1}{\sqrt{1+\epsilon}} = 0.5222$. We find numerically $0.4807 < \Phi_{0,crit} < 0.4808$. As we have $\epsilon < \frac{1}{8}$, $\Phi(t)$ shows no oscillations while having $\Phi_0 = 0.5$ larger than $\Phi_{0,crit}$ ensures its convergence. As for the conformally invariant case, Φ decreases monotonically, with an inflexion though, starting around $\tilde{\Phi} = 0.42817$, while H increases monotonically to $\sqrt{\frac{\Lambda}{3}}$. This behaviour of H persists for $\epsilon \rightarrow 0$, $\epsilon > 0$, in agreement with (31).

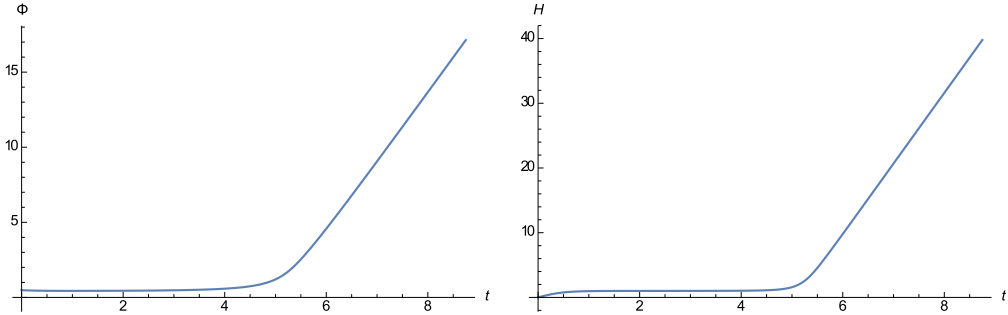


Figure 2: The functions $\Phi(t)$, $H(t)$ are shown for the model parameters of Figure (1) and $\epsilon = 0.1$ but with the initial condition (at the bounce) $\Phi_0 = 0.48 < \Phi_{0,crit}$ so that Φ and H diverge. In sharp contrast to the conformally invariant case, the divergence does not occur in a finite time. Actually, the asymptotic behaviour of both Φ and H is linear in time according to (34), (35).

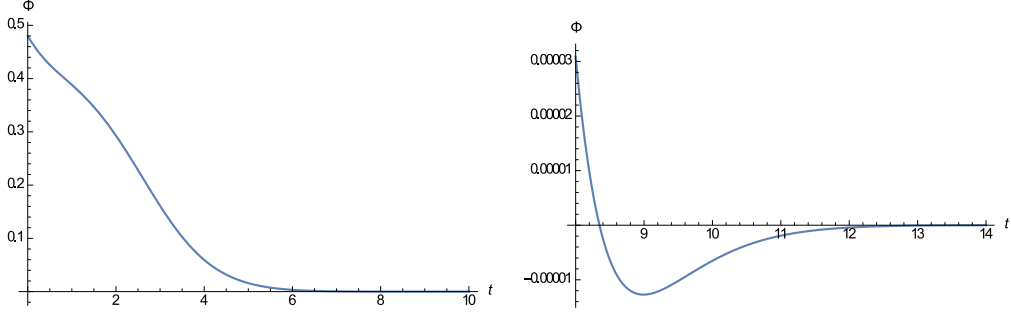


Figure 3: The function $\Phi(t)$ is shown for the model parameters of Figure (1), but now for $\epsilon = 0.25$. We find numerically $0.4737 < \Phi_{0,crit} < 0.4738$. We take here $\Phi_0 = 0.48$ so Φ converges. As we have here $\epsilon > \frac{1}{8}$, oscillations must be present in $\Phi(t)$ at least asymptotically, but these are too small to be seen on the plot. The right panel shows a zoom of the left panel and we can see that $\Phi = 0$ around $t = 8.4$. This is only possible as $\dot{H} < 0$ at the same time. For $\epsilon \neq 0$ one can have $\dot{H} < 0$ but this was not possible for the conformally invariant case.

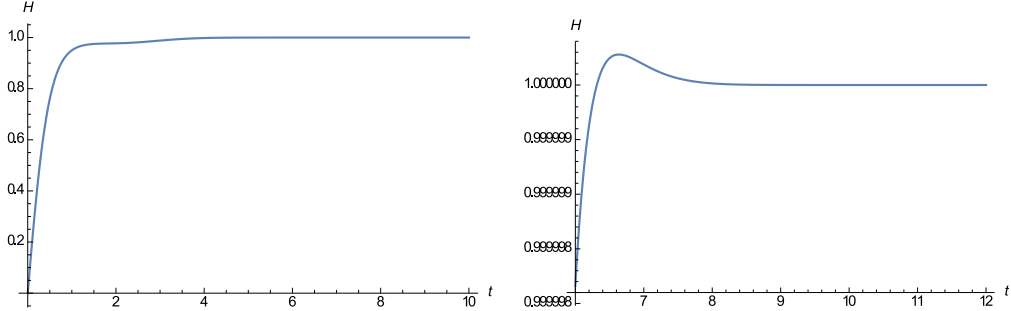


Figure 4: The function $H(t)$ is shown for the model parameters of Figure (1) and $\epsilon = 0.25$. The Hubble parameter H tends to its asymptotic value but the increase is not monotonic. A first maximum appears around $t = 6.5$ as we see on the right panel. Actually another very weak maximum occurs around $t = 12.6$. The asymptotic behaviour satisfies $\dot{H} < 0$. This illustrates the ability of the system to move dynamically from the phantom to the non-phantom regime, such a behaviour however did not occur for $\epsilon = 0$ where a monotonic increase of H was found. As expected, $\dot{H} < 0$ at the time where Φ vanishes, see Figure 3.

5 Numerical simulations

5.1 $\epsilon > 0$

We start first with $\epsilon > 0$. Different cases will arise here for given system parameters Λ , c and κ . Let us take $\epsilon < \epsilon_{max}$ (see (11)) so that a nonvanishing interval of initial values (at $t = 0$) $\Phi_{0,min} < \Phi_0 < \Phi_{max}$ is allowed. As we noted already earlier, this interval of initial values is embedded in the allowed interval for $\epsilon = 0$. By increasing ϵ from zero to ϵ_{max} , this interval will shrink until it vanishes. We find that for any $\epsilon < \epsilon_{max}$ the interval of allowed initial values will contain a critical value $\Phi_{0,cr}$. Then there exists a subinterval $\Phi_{0,cr} < \Phi_0 < \Phi_{max}$ for which Φ tends asymptotically to zero. In that case we get for $\epsilon \leq \frac{1}{8}$ that Φ decreases monotonically to zero like $\Phi^{(0)}$. On the other hand for $\epsilon > \frac{1}{8}$, Φ exhibits damped oscillations around zero in the asymptotic regime. Indeed, let us consider more carefully the nature of the asymptotic behaviour of Φ in this case. As $\Phi \rightarrow 0$ asymptotically, the equation satisfied by Φ in this regime is a very simple generalization of (24), namely

$$\ddot{\Phi} + \sqrt{3\Lambda} \dot{\Phi} + \frac{2\Lambda}{3}(1 + \epsilon)\Phi = 0 , \quad (33)$$

leading to an overdamped oscillation for $\epsilon \leq \frac{1}{8}$. Hence in the neighbourhood of $\Phi^{(0)}$ the asymptotic solution shows no oscillations but this is no longer true for $\epsilon > \frac{1}{8}$. An interesting point is that while the possibility to have damped oscillations is immediate from (33), this is not obvious by inspection of (29) and of the similar equations for all higher order functions $\Phi^{(n)}$. Actually, the oscillations of Φ come from the presence of all the inhomogeneous parts $I^{(n)}$ as the homogeneous part alone leads to an exponentially decaying solution without oscillations. Another interesting point here is the possibility for Φ to vanish and to be negative in contrast to the conformally invariant case. This is because here, \dot{H} can change sign and become negative. This is unlike the conformal case where $\dot{H}^{(0)} > 0$, which implied by inspection of (7) that $\Phi = 0$ is impossible. We found numerically as expected that Φ has zeroes only in regions where $\dot{H} < 0$. Moreover we find $\dot{H} < 0$ in the asymptotic regime for $\epsilon > \frac{1}{8}$ so that an infinite number of zeroes of Φ is possible. Actually $\dot{H} < 0$ can already occur for $\epsilon < \frac{1}{8}$. If we start with $\Phi_0 < \Phi_{0,cr}$, the solution will diverge for $t \rightarrow \infty$. However it is interesting that they do not become singular in a finite time as for $\epsilon = 0$. Instead both Φ and H tend to infinity for $t \rightarrow \infty$. It is not hard to see from equations (6) - (8) that

$$\Phi \sim \alpha t , \quad H \sim \beta t \quad (34)$$

are possible asymptotic solutions for $t \rightarrow \infty$. Their ratio is constant in this regime and we find easily

$$\frac{\alpha}{\beta} = \sqrt{\frac{1 + \epsilon}{2c}} . \quad (35)$$

We find numerically that both slopes go to infinity as ϵ goes to zero. Remember that in the conformal case the system has the remarkable property that the dynamics of Φ decouples essentially from H , more precisely in equation (1) only a first integral of eq.(3) will appear. Hence in that case one could have that Φ diverges, in a finite time even, while H remains perfectly regular. This is no longer true here. Figures 1 - 4 show these different behaviours for a specific choice of the system parameters κ , Λ and c and $\epsilon = 0.1, 0.25$. The qualitative behaviour is similar for other system parameters.

5.2 $\epsilon < 0$

Let us turn now our attention to $\epsilon < 0$. In contrast to the case with $\epsilon > 0$, now the interval of allowed initial values includes the interval for $\epsilon = 0$, so some initial values are possible which were forbidden in the conformally invariant case. It is further clear from eq.(33) which is valid for any sign of ϵ that Φ behaves asymptotically like an overdamped oscillation. A value $\Phi_{0,crit}$ is found again and interestingly we find that $\Phi_{0,crit}$ itself can be larger than $\Phi_{max}^{(0)}$. For $\Phi_0 > \Phi_{0,crit}$, Φ decreases monotonically and tends asymptotically to zero. The Hubble parameter H in turn increases first, before decreasing asymptotically and tending to $\sqrt{\frac{\Lambda}{3}}$. For $\epsilon \rightarrow 0$, H will first reach a maximum before tending to its asymptotic value with $\dot{H} < 0$. This is in perfect agreement with our expression (31): asymptotically, we have $H > H^{(0)}$ and $\dot{H} < 0$ (while of course $H \rightarrow H^{(0)}$ and $\dot{H} \rightarrow 0$).

When $\Phi_0 < \Phi_{0,crit}$, Φ is found to diverge in a finite time. This is similar to the singular behaviour occurring for $\epsilon = 0$. Now however $H \rightarrow -\infty$ in a finite time as well, corresponding to a Big Crunch.

For $\epsilon < 0$ a critical value ϵ_c is also found such that for $|\epsilon| > |\epsilon_c|$ both Φ and H diverge for whatever value Φ_0 in the allowed interval of initial values. However for $\epsilon < 0$ this divergence happens in a finite time, in a way similar for $|\epsilon| < |\epsilon_c|$ with $\Phi_0 < \Phi_{0,crit}$.

Figures 5 - 7 show these various behaviours for specific values of the system parameters. The same qualitative behaviour is obtained for other system parameters.

6 Conclusions

A family of spatially-flat bouncing solutions with a nonzero measure set of initial conditions regular after the bounce (located at $t = 0$) were recently found in the framework of scalar-tensor gravity. In this model, the scalar field was conformally coupled, allowing the derivation of exact analytical expressions. A crucial aspect was the decoupling of the Hubble parameter $H(t)$ from $\Phi(t)$. Indeed $H(t)$ coupled only to a first integral of $\Phi(t)$ and does not depend on the dynamics of $\Phi(t)$. The primary purpose of the present work is to

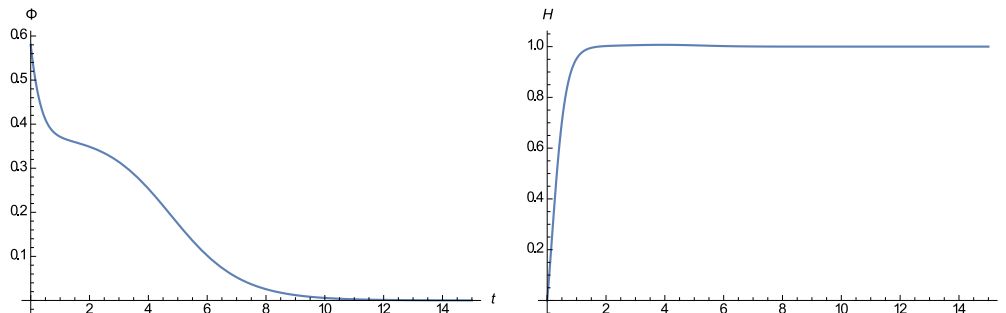


Figure 5: The functions $\Phi(t)$ and $H(t)$ are shown starting for the model parameters of Figure 1 and $\epsilon_c < \epsilon = -0.15$ with $-0.177 < \epsilon_c < -0.176$. For $|\epsilon| > |\epsilon_c|$ both functions would diverge for any initial value Φ_0 . We find $0.559 < \Phi_{0,crit} < 0.560$ and we take $\Phi_0 = 0.58 > \Phi_{0,crit}$. The behaviour is essentially similar to the conformally invariant case, the increase of H however is not monotonic as will be seen on Figure 6.

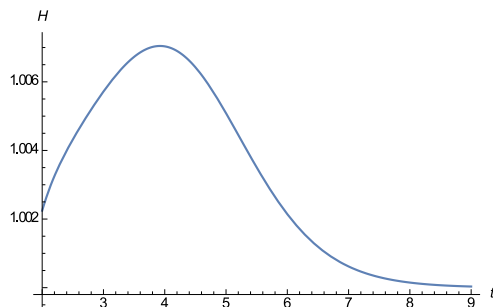


Figure 6: The function $H(t)$ is shown, it is a zoom of Figure 5 showing its nonmonotonic increase. The system evolves dynamically from a phantom to a non phantom regime. Actually this behaviour persists for $\epsilon \rightarrow 0$, $\epsilon < 0$, though the maximum becomes less and less pronounced, hence $H \rightarrow H^{(0)}$ from above with $\dot{H} < 0$, in perfect agreement with eq.(31).

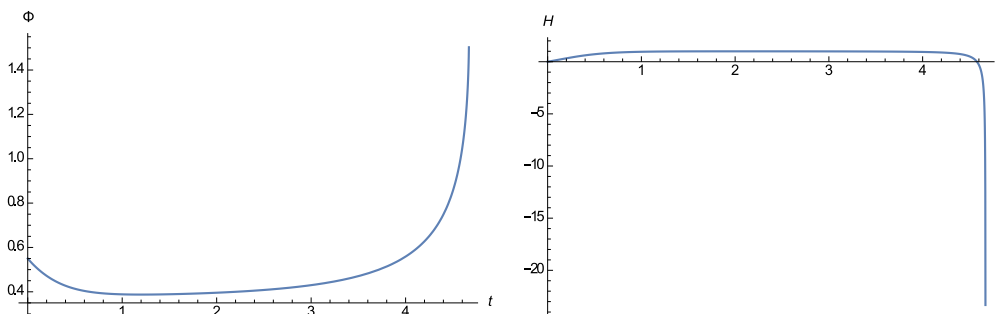


Figure 7: The functions $\Phi(t)$ and $H(t)$ for the model parameters of Figure 1 for $\epsilon = -0.15$ and $\Phi_0 = 0.55 < \Phi_{0,crit}$. Now both Φ and H diverge in a finite time. The divergence of Φ is similar of that found for $\epsilon = 0$. The divergence of H which is impossible for $\epsilon = 0$, represents a contracting universe with a Big Crunch in the unphysical regime $F < 0$.

extend these results and to investigate whether regular bouncing solutions exist also for $\xi \equiv \frac{1}{6}(1 + \epsilon) \neq \frac{1}{6}$.

We have shown that there exists a family of regular bouncing solutions at least in the neighbourhood of the conformally coupled ones, corresponding to $\xi \neq \frac{1}{6}$ arbitrarily close to $\frac{1}{6}$. These scalar field solutions vanish exponentially with damped oscillations and an infinite number of zeroes when $\frac{1}{8} < \epsilon < \epsilon_{max}$, and in a monotonous way for $0 < \epsilon \leq \frac{1}{8}$. Regular non oscillating solutions tending to zero were also found when $\epsilon < 0$. In a systematic expansion of the solutions in terms of ϵ we show explicitly that the solutions Φ and H vanish exponentially at each order. Analytic asymptotic expressions were also given at first order.

Proceeding with a numerical study, all the above properties were confirmed also for significant departures from $\epsilon = 0$. Like in the conformally coupled case, whenever regular solutions exist we find that part of the allowed initial interval with $\Phi_0 > \Phi_{0,cr}$ will lead to regular solutions, while the other initial values will lead to divergent solutions.

For $\epsilon > 0$ we find that regular bouncing solutions exist for any $0 < \epsilon < \epsilon_{max}$. As $\epsilon \rightarrow \epsilon_{max}$ and hence $\Phi_{max} \rightarrow \Phi_{0,min}$, we always find a critical initial value $\Phi_{0,min} < \Phi_{0,crit} < \Phi_{max}$. The divergence of both Φ and H for $\Phi_0 < \Phi_{0,crit}$ is linear in time for $t \rightarrow \infty$ in contrast with the conformally coupled case where Φ diverges in a finite time. Here too it implies in particular that the model becomes unviable in a finite time when F becomes negative.

When $\epsilon < 0$, a critical value ϵ_c was found such that all solutions diverge for $\epsilon < \epsilon_c$. For $\epsilon_c < \epsilon < 0$ however, a critical initial value $\Phi_{0,min} < \Phi_{0,crit} < \Phi_{max}$ and regular solutions for $\Phi_0 > \Phi_{0,crit}$ are found. In this case, the nonregular solutions diverge in a finite time with the occurrence of a Big Crunch.

Another aspect of our work is related to the metric dynamics. As for the conformally invariant case, all regular bouncing solutions tend asymptotically to a de Sitter space with Hubble parameter $\sqrt{\frac{\Lambda}{3}}$ while gravity tends to GR. However in contrast to the conformally coupled case where $H(t)$ does not depend on $\Phi^{(0)}(t)$ (for given parameters of the system) – a remarkable property allowing a complete integration of the problem – this is no longer the case when $\epsilon \neq 0$. To incorporate this bouncing model in a realistic cosmology, some dynamical evolution into a decelerated stage after the bounce should take place. Though we find various cases where \dot{H} becomes negative, this change is not significant enough and more drastic departures from conformal invariance are presumably needed.

References

- [1] A. A. Starobinsky, Sov. Astron. Lett. **4**, 82 (1978).

- [2] D. N. Page, *Class. Quantum Grav.* **1**, 417 (1984).
- [3] A. Yu. Kamenshchik, I. M. Khalatnikov, A. V. Toporensky, *Int. J. Mod. Phys. D* **7**, 673 (1997) [gr-qc/9801064].
- [4] G. W. Gibbons, N. Turok, *Phys. Rev. D* **77**, 063516 (2008) [hep-th/0609095].
- [5] D. Tretyakova, A. Shatskij, I. Novikov, S. Alexeyev, *Phys. Rev. D* **85**, 124059 (2012) [arXiv:1112.3770].
- [6] J. D. Barrow, D. Sloan, *Phys. Rev. D* **88**, 023518 (2013) [arXiv:1304.6699].
- [7] A. Ashtekar, T. Pawłowski, P. Singh, *Phys. Rev. D* **74**, 084003 (2006) [gr-qc/0607039].
- [8] T. Qiu, J. Evslin, Y. F. Cai, M. Li, X. Zhang, *JCAP* **1110**, 036 (2011) [arXiv:1108.0593].
- [9] D. A. Easson, I. Sawicki, A. Vikman, *JCAP* **1111**, 021 (2011) [arXiv:1109.1047].
- [10] Y. F. Cai, D. A. Easson, R. Brandenberger, *JCAP* **1208**, 020 (2012) [arXiv:1206.2382].
- [11] D. Battefeld, P. Peter, *Phys. Rept.* **571**, 1 (2015) [arXiv:1406.2790].
- [12] P. Peter, R. Brandenberger, [arXiv:1603.05834].
- [13] Planck Collaboration, P.A.R. Ade et al, *Astron. Astrophys.* **571**, A16 (2014) [arXiv:1303.5076].
- [14] B. Boisseau, G. Esposito-Farèse, D. Polarski, A. A. Starobinsky, *Phys. Rev. Lett.* **85**, 2236 (2000) [gr-qc/0001066].
- [15] D. Torres, *Phys. Rev. D* **66**, 043522 (2002).
- [16] R. Gannouji, D. Polarski, A. Ranquet, A. A. Starobinsky, *JCAP* **0609**, 016 (2006) [astro-ph/0606287].
- [17] B. Boisseau, H. Giacomini, D. Polarski, A. A. Starobinsky, *JCAP* **0715**, 002 (2015) [arXiv:1504.07927].
- [18] B. Boisseau, H. Giacomini, D. Polarski, *JCAP* **1015**, 033 (2015) [arXiv:1507.00792].
- [19] I. Bars, S.-H. Chen, P. J. Steinhardt, N. Turok, *Phys.Rev. D* **86** 083542 (2012) [arXiv:1207.1940].
- [20] A. Yu. Kamenshchik, E. O. Pozdeeva, A. Tronconi, G. Venturi, S. Yu. Vernov, *Class. Quant. Grav.* **33** no.1, 015004 (2016) [arXiv:1509.00590].

- [21] G. 't Hooft, arXiv:1511.04427.
- [22] D. Boyanovsky, arXiv:1602.05609.
- [23] V. A. Rubakov, JCAP **0909**, 030 (2009) [arXiv:0906.3693].
- [24] C. Martinez, R. Troncoso, J. Zanelli, Phys. Rev. D**67**, 024008 (2003) [hep-th/0205319].
- [25] S. de Haro, I. Papadimitriou, A. C. Petkou, Phys. Rev. Lett. **98**, 231601 (2007) [hep-th/0611315].
- [26] O. Hrycyna, arXiv:1511.08736.
- [27] G. Esposito-Farèse, D. Polarski, Phys. Rev. D **63**, 063504 (2001) [gr-qc/0009034].
- [28] L. M. Perko, Differential equations and dynamical systems. Third edition. Texts in Applied Mathematics, 7. Springer-Verlag, New York, 2001.